

Lyapunov Redesign of Analog Phase-Lock Loops

DANIEL Y. ABRAMOVITCH*
Hewlett-Packard Laboratories
 1501 Page Mill Road, MS: 2U
 Palo Alto, CA 94304
 415-857-3806
 E-mail: danny@hpldya.hpl.hp.com

Abstract— In general, the design of phase-lock loops has been done by a combination of linear analysis, phase plane plots, rule of thumb, and simulation. Very few analytical tools have been used to determine the stability of the nonlinear models used for these devices. A method from the control literature known as Lyapunov redesign[1] has recently been used to design a third order phase-lock loop whose nonlinear model is guaranteed to be stable [2]. In this paper, this technique is demonstrated to be an effective stability analysis and design technique for many analog phase-lock loops. The ability of loops designed using these techniques to track a phase step is also proven.

analysis and design of phase-lock loops has been done by a combination of linear analysis, phase plane plots, rule of thumb, and simulation[3, 4, 5, 6, 7]. A method from the control literature known as Lyapunov redesign[1] has recently been used to design a third order phase-lock loop whose nonlinear model is guaranteed to be stable [2]. In this paper, this technique is demonstrated to be an effective stability analysis and design technique for many analog phase-lock loops.

Typically, stability analysis deals with noise free models. This will be the case here. While practical analysis and simulation of a real system must include some noise model, a prerequisite for such analysis is either the knowledge or the assumption that the loop is stable. This paper will attempt to provide the former. Furthermore, stability analysis deals with the homogeneous (no input) differential equation of the system. This will be useful here, as it is often convenient to perform block diagram manipulations to give a set of states for which it is easy to extract stability results. Finally, stability is an asymptotic property. Knowing that a system is stable or that it tracks a step input does not in itself yield performance results. The purpose of this paper is to provide a design method for analog PLLs that will guarantee stability and tracking.

The structure of this paper is as follows. Section II will review the model of an analog PLL. Section III will introduce the necessary definitions and theorems for doing analysis. The actual analysis as applied to phase-lock loops will be done in Section IV.

I. Introduction

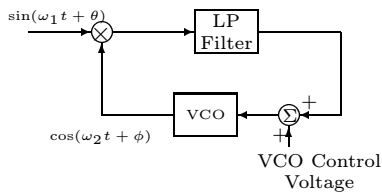


Figure 1: Phase Lock Loop

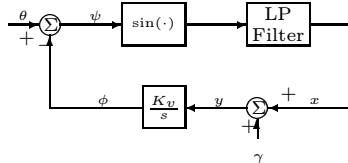


Figure 2: New Model of PLL

Analog phase-lock loops have been around for many years as noted in [3] and [4]. While the field is considered quite mature and there have been many books written on the subject *e.g.*, [5, 6, 7], very little has been said in the commonly available literature about the stability analysis of the nonlinear model of the analog phase-lock loop. There have been a few examples of authors using Lyapunov analysis on PLLs [8, 9, 10], but in general, the

*This work was done while the author was employed by Ford Aerospace in Palo Alto, CA.

II. The Model of an Analog PLL

The block diagram of an analog phase-lock loop (PLL)¹ is shown in Figure 1. While this block diagram is close to the actual implementation of the PLL, several steps are normally taken to facilitate analysis. Using the familiar trigonometric identity in terms of the PLL:

$$2 \sin(\omega_1 t + \theta) \cos(\omega_2 t + \phi) = \sin((\omega_1 + \omega_2)t + \theta + \phi) + \sin((\omega_1 - \omega_2)t + \theta - \phi)(1)$$

¹Digital phase-lock loops are denoted by DPLL.

and then making two fundamental assumptions leads to the commonly used model of the analog PLL. These assumptions are:

1. The first term in (1) is attenuated by the low pass filter.
2. $\omega_1 \approx \omega_2$, so that the difference can be incorporated into ϕ . This means that the VCO can be modeled as an integrator.

Making these assumptions leads to the PLL model shown in Figure 2.

The problem is that this is still a nonlinear system, and as such is in general difficult to analyze. The typical methods of analysis include:

1. Linearization: For ψ small and slowly varying

$$\sin \psi \approx \psi, \quad \cos \psi \approx 1, \quad \text{and } \dot{\psi}^2 \approx 0.$$

While this is useful for studying loops that are near lock, it does not help for analyzing the loop when ψ is large.

2. Phase plane portraits. The disadvantage to this is that phase plane portraits can only completely describe first and second order systems.
3. Simulation.

In this paper, the second method of Lyapunov will be proposed as a method for analyzing the stability and tracking of the *nonlinear* phase lock loop model. It will be shown that a fairly regular procedure known as *Lyapunov redesign*[1], which is quite well known in the area nonlinear control theory is quite applicable to solving this class of problems.

III. Lyapunov Stability

The second method of Lyapunov[11] is commonly used in stability analysis of nonlinear differential equations because it does not require the solution to the differential equation. A very intuitive discussion of this can be found in Ogata [12]. The second method of Lyapunov is based on the generalized energy in the system. If an energy like function of the system state (*i.e.*, a positive definite function of the state which is nonvanishing as long as the state is nonzero) is found which is constantly decreasing, then the system is asymptotically stable. A general form of a vector differential equation is:

$$\dot{x} = f(x, t) \quad \text{where } x, \dot{x} \in R^n. \quad (2)$$

An equilibrium state is any state such that

$$f(x_e, t) = 0. \quad (3)$$

Usually, a transformation is made so that the origin of state space is an equilibrium state *i.e.*,

$$f(0, t) = 0. \quad (4)$$

Theorem 1 (LaSalle's Theorem) [11] *For the system defined by Equation 2, suppose there exists a positive definite scalar function of x , $V(x)$, such that $\dot{V}(x)$ is negative semi-definite *i.e.*,*

$$\begin{aligned} V(x) &> 0, & \dot{V}(x) &\leq 0 & \forall x \neq 0 \\ V(x) &= 0, & \dot{V}(x) &= 0 & x = 0. \end{aligned} \quad (5)$$

Suppose also that the only solution of $\dot{x} = f(x, t)$, $\dot{V}(x) = 0$ is $x(t) = 0$ for all $t \geq 0$. Then $\dot{x} = f(x, t)$ is globally asymptotically stable.

This theorem will prove to be quite useful in the next section. In general V is known as a *Lyapunov function* if it satisfies either LaSalle's Theorem or Lyapunov's Main Stability Theorem [11]. It turns out in practice that Theorem 1 is often easier to satisfy. Another definition that is necessary is that of a sector nonlinearity.

Definition 1 (Sector Nonlinearity) *The function $\phi(\cdot, \cdot)$ is said to belong to sector $[\alpha, \beta]$ if*

$$\alpha y^2 \leq y\phi(t, y) \leq \beta y^2 \quad \forall y \in R, \quad \forall t \geq 0.$$

In other words, a sector nonlinearity would belong to sector $[\alpha, \beta]$ if it fell in the dotted region of Figure 3.

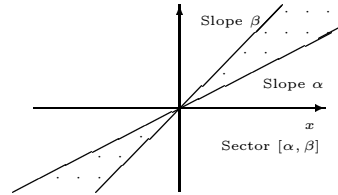


Figure 3: Sector nonlinearity: $\phi \in [\alpha, \beta]$

Lyapunov redesign starts with a candidate Lyapunov function. The function is parameterized by the design parameters of the system in question. These parameters are then chosen so that the candidate Lyapunov function meets the requirements of either Lyapunov's Theorem or LaSalle's Theorem. In this paper, that function will have the form

$$V = \int_0^z f(\sigma) d\sigma + x^T P x \quad (6)$$

which was introduced by LaSalle and Lefschetz[13]. P is a positive definite matrix, x is some portion of the system state, and $f(\cdot)$ is a nonlinearity which lies in sector $[0, \infty]$. That is to say

$$0 \leq f(\sigma)\sigma. \quad (7)$$

The key is to satisfy conditions such that $V \geq 0$, but

$$\dot{V} = f(z)\dot{z} + x^T P \dot{x} \leq 0. \quad (8)$$

In the case of the analog PLL, $f(\sigma) = \sin(\sigma)$ and it is fairly easy to see that this is in sector $[0, \infty]$ for $-\pi < \sigma < \pi$. Also, in the cases when P is a 2×2 matrix (third order PLLs) the conditions for $P > 0$ are

$$p_{11} > 0, \quad p_{11}p_{22} > p_{12}^2, \quad \implies \quad p_{22} > 0. \quad (9)$$

IV. Nonlinear Analysis of PLLs

In this section, the above Lyapunov analysis techniques are applied to a variety of phase-lock loops. The structure of this section will be a series of examples of second and third order phase-lock loops, some with zeros and some without. In each case, the differential equations and a corresponding block diagram will be shown. A Lyapunov function is chosen which is parameterized by the parameters of the analog phase-lock loop. Conditions for stability of the PLL are then determined in terms of the loop parameters. Finally, using these same loop parameters, the tracking of step inputs to the PLL is proven.

IV.A Second Order PLL with No Zeros

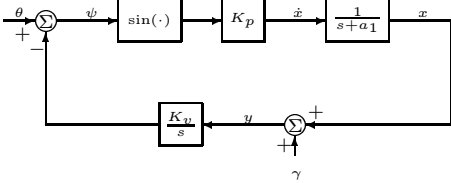


Figure 4: Second Order Phase-Lock Loop with No Zeros

A second order PLL with no zeros is shown in Figure 4. The differential equations corresponding to this loop with no external inputs ($\theta = \gamma = 0$) are:

$$\dot{x} = K_p \sin \psi - a_1 x \text{ and} \quad (10)$$

$$\dot{\psi} = -K_v(x + \gamma) = -K_v x, \quad (11)$$

where $\psi = \theta - \phi = -\phi$. Choose

$$V = \int_0^\psi \sin(\sigma) d\sigma + \frac{1}{2} p x^2, \quad p > 0. \quad (12)$$

The term under the integral is positive for $-\pi < \psi < \pi$ and this fact will be used quite often. Then

$$\dot{V} = \sin(\psi) \dot{\psi} + p x \dot{x} \quad (13)$$

$$= p x (K_p \sin \psi - a_1 x) + \sin \psi (-K_v x) \quad (14)$$

$$= -a_1 p x^2 + \sin(\psi) x (p K_p - K_v). \quad (15)$$

In order to invoke LaSalle's Theorem, we must have $V(\psi, x) \geq 0$ with $V = 0 \iff \psi = x = 0$ and $\dot{V} \leq 0$. Assuming $\psi \in (-\pi, \pi)$ the condition for $V \geq 0$ is

$$p > 0. \quad (16)$$

The conditions that guarantee $\dot{V} \leq 0$ are:

$$a_1 p > 0, \text{ and} \quad (17)$$

$$p K_p - K_v = 0. \quad (18)$$

For $K_p, K_v > 0$ it is always possible to satisfy Conditions 18 and 16 by picking

$$p = \frac{K_v}{K_p}. \quad (19)$$

Condition 17 is easy to satisfy by picking

$$a_1 > 0, \quad (20)$$

which leaves

$$V = \int_0^\psi \sin(\sigma) d\sigma + \frac{1}{2} \frac{K_v}{K_p} x^2 \text{ and} \quad (21)$$

$$\dot{V} = -a_1 \frac{K_v}{K_p} x^2 \leq 0. \quad (22)$$

Finally, the only values for ψ and x which results in $\dot{V} = \dot{x} = \dot{\psi} = 0$ is $\psi = x = 0$.

IV.A.1 Tracking a Phase Step

The second order PLL with no zeros designed above is stable. It will now be shown that this loop can also track a step input. The equations for the PLL corresponding to Figure 4 with an input at θ are:

$$\dot{x} = K_p \sin \psi - a_1 x, \quad (23)$$

$$\psi = \theta - \phi, \text{ and} \quad (24)$$

$$\dot{\psi} = \dot{\theta} - K_v(x + \gamma) = \dot{\theta} - K_v x. \quad (25)$$

As above, choose the Lyapunov function

$$V = \int_0^\psi \sin(\sigma) d\sigma + \frac{1}{2} \frac{K_v}{K_p} x^2, \quad (26)$$

where the choice of p from Equation (19) has as been made and $a_1 > 0$. Then

$$\dot{V} = \sin(\psi) \dot{\theta} - a_1 \frac{K_v}{K_p} x^2, \quad (27)$$

where the second term is the same as the no input case, Equation (22), and the first term corresponds to excitation caused by the input to θ . Now say θ is a step. Then

$$\dot{\theta} = \theta_0 \delta(t), \quad (28)$$

where $\delta(t)$ is the impulse function. Integrating (27) forward in time and noting that $\sin \psi(0^+) = \sin \theta_0$ yields (due to the sifting property of the impulse function):

$$V(t) = \int_{0^-}^t \sin(\psi) \dot{\theta} dt - a_1 \frac{K_v}{K_p} \int_{0^-}^t x^2 dt \quad (29)$$

$$= \theta_0 \sin \theta_0 - a_1 \frac{K_v}{K_p} \int_{0^-}^t x^2 dt \quad (30)$$

The first term of (30) is a positive constant for $-\pi < \theta_0 < \pi$. The second term is a negative number which will grow without bound unless x goes to 0. If x did not converge to 0, then $V(t)$ would eventually become negative which is impossible since $V(t)$ was chosen to be a positive definite function. Thus, x must converge to 0. As in the discussion of stability, the only value of ψ for which x can remain identically 0 is $\psi = 0$, thus the loop must track a phase step input.

IV.B Second Order PLL with One Zero

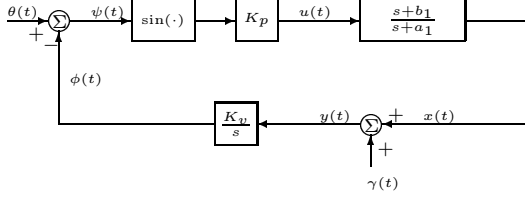


Figure 5: Second Order Phase-Lock Loop with 1 Zero

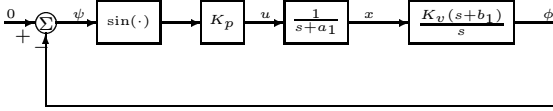


Figure 6: Closed-Loop Nonlinear System

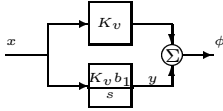


Figure 7: Drawing out the state variables

A second order PLL with one zero is shown in Figure 5. In this case, the analysis is simplified a bit if the loop is redrawn as in Figure 6. These two loops are equivalent from the point of view of closed-loop stability. The second configuration merely makes it possible to draw some new state variables out of the block from x to ϕ as shown in Figure 7. The differential equations corresponding to this loop with no external inputs ($\theta = \gamma = 0$) are:

$$\dot{y} = K_v b_1 x \quad (31)$$

$$\phi = K_v x + y \quad (32)$$

$$\dot{x} = K_p \sin \psi - a_1 x \quad (33)$$

$$\dot{\phi} = K_v K_p \sin \psi + K_v b_1 x - K_v a_1 x \quad (34)$$

$$\dot{\psi} = -K_v K_p \sin \psi - K_v b_1 x + K_v a_1 x \quad (35)$$

where $\psi = \theta - \phi = -\phi$. Choose

$$V = \int_0^\psi \sin(\sigma) d\sigma + \frac{1}{2} p x^2, \quad p > 0. \quad (36)$$

Using analysis entirely analogous to that in Section IV.A and choosing $K_p, K_v > 0$ the conditions for satisfying

LaSalle's Theorem are:

$$p = \frac{K_v(b_1 - a_1)}{K_p} \quad \text{and} \quad (37)$$

$$b_1 > a_1 > 0. \quad (38)$$

This leaves

$$V = \int_0^\psi \sin(\sigma) d\sigma + \frac{1}{2} \frac{K_v(b_1 - a_1)}{K_p} x^2 \quad \text{and} \quad (39)$$

$$\dot{V} = -K_v K_p [\sin \psi]^2 - a_1 \frac{K_v(b_1 - a_1)}{K_p} x^2 \leq 0. \quad (40)$$

Finally, the only values for ψ and x which results in $\dot{V} = \dot{x} = \dot{\psi} = 0$ is $\psi = x = 0$.

It is interesting to note that if we chose $b_1 = a_1$ then the pole and zero from x to ϕ would cancel leaving $x = \phi$. That is to say, the model can be reduced to that of a first order analog phase-lock loop. For this new model, this same Lyapunov function works, except that $p = 0$ and thus

$$V = \int_0^\psi \sin(\sigma) d\sigma \quad \text{and} \quad (41)$$

$$\dot{V} = -K_v K_p [\sin \psi]^2 \leq 0. \quad (42)$$

IV.B.1 Tracking a Phase Step

It turns out that a completely analogous result for tracking holds for the second order PLL with one zero as was shown for the second order PLL with no zeros. As above, choose V from (36) subject to (37) and (38) to yield

$$\dot{V} = \sin(\psi) \dot{\theta} - K_v K_p [\sin \psi]^2 - a_1 \frac{K_v(b_1 - a_1)}{K_p} x^2, \quad (43)$$

where the last two terms are the same as the no input case, Equation (40), and the first term corresponds to excitation caused by the input to θ . As before θ is a step, so integrating (43) forward in time yields (see (29)–(30) for details):

$$V(t) = \theta_0 \sin \theta_0 - K_v K_p \int_{0^-}^t [\sin \psi]^2 dt - a_1 \frac{K_v(b_1 - a_1)}{K_p} \int_{0^-}^t x^2 dt \quad (44)$$

The first term of (44) is a positive constant for $-\pi < \theta_0 < \pi$. The latter terms are negative numbers which will grow without bound unless ψ and x go to 0. As before, we reach the conclusion that ψ and x must go to 0 to preserve the positive definiteness of V and thus this loop tracks a step input to θ .

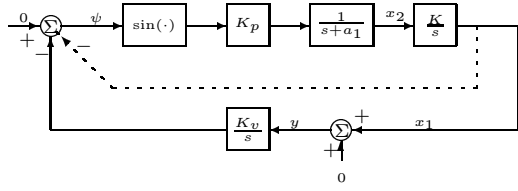


Figure 8: Third Order Phase-Lock Loop with No Zeros. Dashed line is velocity feedback which does not correspond to a PLL.

IV.C Third Order PLL with No Zeros

A third order phase-lock loop with no zeros turns out to be unstable, even in the linearized (small signal) model. It turns out that the feedback structure can be stabilized by adding what is known in the controls literature as *velocity feedback*² as shown in with the dashed line in Figure 8. While this is useful from a theoretical point of view and in fact a Lyapunov function can be found for this system, this model no longer corresponds to a phase-lock loop and will not be pursued here.

IV.D Third Order PLL with One Zero

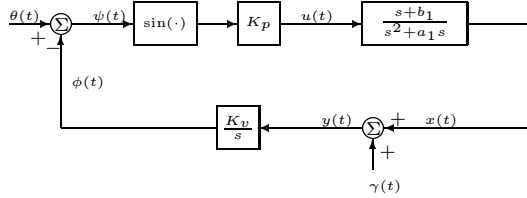


Figure 9: Third Order Phase-Lock Loop with 1 Zero

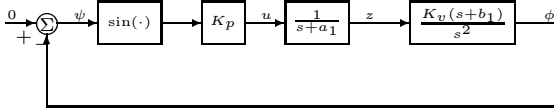


Figure 10: Closed-Loop Nonlinear System

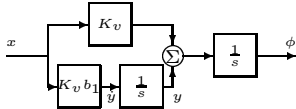


Figure 11: Drawing out the state variables

A third order analog PLL with one zero is shown in Figure 9. Again, for the sake of stability and tracking analysis it is convenient to redraw the loop as in Figure 10. From here the necessary state variables can be drawn out as in Figure 11. The state equations corresponding to Figures 10 and 11 are:

$$\dot{x} = K_p \sin \psi - a_1 z, \quad (45)$$

$$\dot{y} = K_v b_1 x, \quad \text{and} \quad (46)$$

$$\dot{\psi} = -\dot{\phi} = -K_v x - y. \quad (47)$$

²In a mechanical analog of the analog PLL, ϕ would correspond to the position and x would correspond to the velocity.

Choose

$$V = \int_0^\psi \sin(\sigma) d\sigma + \frac{1}{2} \begin{bmatrix} x & y \end{bmatrix} P \begin{bmatrix} x \\ y \end{bmatrix}, \quad (48)$$

where P is a symmetric, positive definite, 2×2 matrix. In order to invoke LaSalle's Theorem, we must have $V(\psi, x, y) \geq 0$ with $V = 0 \iff \psi = x = y = 0$ and $\dot{V} \leq 0$. Assuming $\psi \in (-\pi, \pi)$ we can satisfy (9) with

$$P = \begin{bmatrix} \frac{K_v}{K_p} & \frac{1}{K_p} \\ \frac{1}{K_p} & \frac{a_1}{K_p K_v b_1} \end{bmatrix}, \quad (49)$$

$$a_1 > b_1 > 0, \quad \text{and} \quad (50)$$

$$K_p, K_v > 0. \quad (51)$$

Thus, $V \geq 0$ and

$$\dot{V} = -x^2 \left[\frac{K_v}{K_p} (a_1 - b_1) \right] \leq 0. \quad (52)$$

Finally, the only place that \dot{V} and (53) – (55) can vanish is for $z = y = \phi = \psi = 0$, so using LaSalle's Theorem proves stability.

IV.E Third Order PLL with Two Zeros

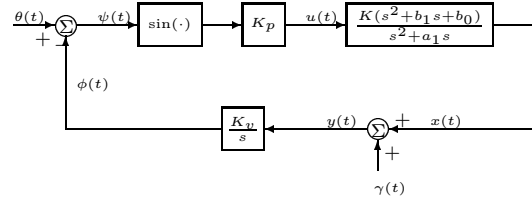


Figure 12: Third Order Phase-Lock Loop with 2 Zeros

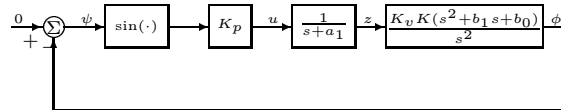


Figure 13: Closed-Loop Nonlinear System

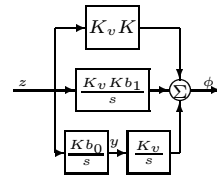


Figure 14: Drawing out the state variables

The block diagram of the PLL that was analyzed in full [2] is shown in Figure 12. The results will be summarized here. This is a third order PLL with two zeros. Note that an extra gain, K , is involve in this model, but for consistency with the previous examples this could be set to 1 without loss of generality. For the sake of stability and tracking analysis it is convenient to redraw

the loop as in Figure 13. From here the necessary state variables can be drawn out as in Figure 14. The state equations corresponding to Figures 13 and 14 are:

$$\dot{z} = K_p \sin \psi - a_1 z, \quad (53)$$

$$\dot{y} = K b_0 z, \text{ and} \quad (54)$$

$$\dot{\psi} = -K_v K K_p \sin \psi - K_v K (b_1 - a_1) z - K_v y. \quad (55)$$

Choose

$$V = \int_0^\psi \sin(\sigma) d\sigma + \frac{1}{2} \begin{bmatrix} z & y \end{bmatrix} P \begin{bmatrix} z \\ y \end{bmatrix}, \quad (56)$$

where P is a symmetric, positive definite, 2×2 matrix. In order to invoke LaSalle's Theorem, we must have $V(\psi, y, z) \geq 0$ with $V = 0 \iff \psi = y = z = 0$ and $\dot{V} \leq 0$. Assuming $\psi \in (-\pi, \pi)$ we can satisfy (9) and guarantee $V \geq 0$ and $\dot{V} \leq 0$ with:

$$P = \begin{bmatrix} \frac{K_v K}{K_p} (b_1 - a_1) & \frac{K_v}{K_p} \\ \frac{K_v}{K_p} & \frac{K_v a_1}{K_p K b_0} \end{bmatrix}, \quad (57)$$

$$K_v K K_p > 0 \iff \frac{K_v K}{K_p} > 0, \quad K_p \neq 0 \quad (58)$$

$$b_1 > a_1, \text{ (} b_1, a_1 \text{ same sign) and} \quad (59)$$

$$b_0 - (b_1 - a_1) a_1 < 0. \quad (60)$$

It is convenient to choose both b_0 and $a_1 > 0$ since this corresponds to a stable filter. Also, it is convenient to choose K_p , K_v , and K positive, leaving

$$\dot{V} = -\sin^2 \psi [K_v K K_p] + z^2 \left[\frac{K_v K}{K_p} (b_0 - (b_1 - a_1) a_1) \right] \leq 0. \quad (61)$$

Finally, the only place that \dot{V} and (45) – (47) can vanish is for $x = y = \phi = \psi = 0$, so using LaSalle's Theorem proves stability.

IV.F Tracking for Third Order Loops

The tracking analysis of a step input for the third order loops is completely analogous to that of the second order loops and will be omitted here for brevity.

V. Discussion

It is interesting to note that the conditions for stability of the nonlinear second order models *i.e.*, $K_v K_p > 0$, $a_1 > 0$, and (in the case of a zero) $b_1 > a_1$ correspond to the linear design rules. First $K_v K_p > 0$ guarantees negative feedback for the loop. The conditions on a_1 and b_1 (if present) correspond to the filter being a stable, low-pass filter. As the presence of a stable, low-pass filter is always assumed in the design of a PLL, this implies that all second order PLLs that are designed using the standard assumption in Section II are stable. Thus, the theory here reinforces what has been empirically observed by many designers over the years.

VI. Conclusions

This paper has endeavored to approach the problem of stability and tracking analysis for the nonlinear model of analog phase-lock loops in a novel manner. The theory used, Lyapunov analysis, and the particular method for design, Lyapunov redesign, are not new to control theory. Only their application to analog phase-lock loops is new. This method appears to be quite useful for this problem and its applicability seems to be quite broad.

References

- [1] P. C. Parks, "Liapunov redesign of model reference adaptive control systems," *IEEE Trans. on Automatic Control*, vol. AC-11, July 1966.
- [2] D. Y. Abramovitch, "Analysis and design of a third order phase-lock loop," in *Proceedings of the IEEE Military Communications Conference*, IEEE, October 1988.
- [3] S. C. Gupta, "Phase-locked loops," *Proceedings of the IEEE*, vol. 63, pp. 291–306, February 1975.
- [4] W. C. Lindsey and C. M. Chie, "A survey of digital phase-locked loops," *Proceedings of the IEEE*, vol. 69, pp. 410–431, April 1981.
- [5] F. M. Gardner, *Phaselock Techniques*. New York, NY: John Wiley & Sons, second ed., 1979.
- [6] A. Blanchard, *Phase-Locked Loops*. New York, NY: John Wiley & Sons, 1976.
- [7] A. J. Viterbi, *Principles of Coherent Communication*. McGraw-Hill Series in Systems Science, New York, NY: McGraw-Hill, 1966.
- [8] J. L. Willems, "Acquisition conditions for phase-lock loops," *Int. J. Electronics*, vol. 26, no. 2, pp. 137–144, 1969.
- [9] G. D. Maria and L. Sciavicco, "On the dynamic behavior of phase-locked loops," *Ricerche de Automatica*, vol. 11, pp. 1–17, June 1980.
- [10] S. M. Yousif, "Liapunov stability of nonlinear phase-locked loop," in *IEEE 1979 Region VI Conference Record*, April 1979.
- [11] R. E. Kalman and J. E. Bertram, "Control system analysis and design via the "Second Method" of Lyapunov, Part 1: Continuous-Time Systems," *Transactions of the ASME*, 1959.
- [12] K. Ogata, *Modern Control Engineering*. Prentice-Hall Instrumentation and Controls Series, Englewood Cliffs, New Jersey: Prentice-Hall, 1970.
- [13] J. LaSalle and S. Lefschetz, *Stability by Liapunov's Direct Method*. New York, N.Y.: Academic Press, 1961.