Analysis and Design of a Third Order Phase-Lock Loop

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***Abstract*— Typical implementations of a phase-lock loop (PLL) are second order. In this paper we examine the advantages and disadvantages of a third order phase-lock loop. Among the advantages is more design freedom which can result in superior noise rejection and lower steady-state error than a second order PLL. Chief among the disadvantages of the third order PLL is the difficulty of analyzing its stability in the region of nonlinear operation. In this paper we will treat the third order PLL as a nonlinear control system: first examining the small signal (linear) operation and then extending the analysis to the nonlinear region. A useful set of tools from the nonlinear control system world, the second method of Lyapunov and LaSalle's Theorem, will be used to derive stability conditions for the nonlinear model.**

I. Introduction

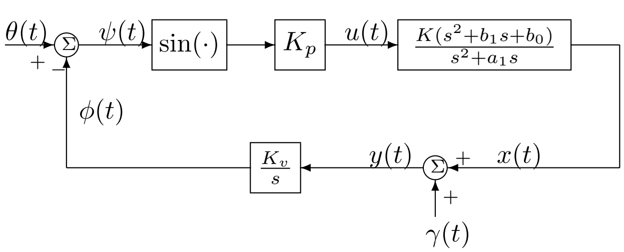

Fig. 1. Third Order Phase-Lock Loop Block Diagram

Typical implementations of a phase-lock loop (PLL) are second order [1], [2], [3], [4]. In this paper we examine the advantages and disadvantages of a third order phase-lock loop. Among the advantages is more design freedom which can result in superior noise rejection and lower steady-state error than a second order PLL. Chief among the disadvantages of the third order PLL is the difficulty of analyzing its stability in the region of nonlinear operation. The differential equations for the third order phase-lock loop in Figure 1 are[1]:

$$\frac{d\phi}{dt} = K_v y(t), \tag{1}$$
$$\psi(t) = \theta(t) - \phi(t), \tag{2}$$
$$u(t) = K_p \sin\psi(t), \text{ and} \tag{3}$$
$$\frac{d^2x}{dt^2} + a_1\frac{dx}{dt} = K\left[\frac{d^2u}{dt^2} + b_1\frac{du}{dt} + b_0 u(t)\right] \tag{4}$$

Equation (1) represents the VCO, (2) defines the error function, (3) is the mixer, and (4) defines the filter. Here $\theta(t)$ and $\gamma(t)$ are external inputs to the system where $\theta(t)$ is the input phase, $\gamma(t)$ is the VCO noise, $\psi(t)$ is the phase error, and $\phi(t)$ is the system output. The other variables, $u(t)$, $x(t)$, and $y(t)$ are merely internal variables. Define $q \triangleq \frac{dy}{dt}$. Then some tedious, but straightforward algebra leads to the following system of differential equations:

$$\frac{d\psi}{dt} = -K_v y(t) + \frac{d\theta}{dt} \tag{5}$$
$$\frac{dy}{dt} = q(t) \tag{6}$$
$$\begin{aligned}\frac{dq}{dt} = {} & KK_p b_0 \sin\psi(t) - \\ & KK_pK_v\left\{\left(K_v y - 2\frac{d\theta}{dt}\right)\sin\psi(t) + b_1\cos\psi(t)\right\}y(t) \\ & -\{a_1 + KK_pK_v\cos\psi(t)\}\,q(t) \\ & +KK_p\left\{\frac{d^2\theta}{dt^2}\cos\psi(t) - \left(\frac{d\theta}{dt}\right)^2\sin\psi(t)\right. \\ & \left. + b_1\frac{d\theta}{dt}\cos\psi(t)\right\} + \frac{d^2\gamma}{dt^2} + a_1\frac{d\gamma}{dt}.\end{aligned} \tag{7}$$

If we assume that the system is initially quiescent , *i.e.*, $\psi(0^-) = y(0^-) = q(0^-) = x(0^-) = 0$, and that neither $\theta(t)$ nor $\gamma(t)$ are impulsive, then the initial conditions can be derived in a tedious but straightforward fashion. These are:

$$\psi(0^+) = \theta(0^+) = \theta_0, \tag{8}$$
$$x(0^+) = KK_p\sin\theta_0, \tag{9}$$
$$y(0^+) = \gamma(0^+) + KK_p\sin\theta_0, \text{ and} \tag{10}$$

[1]The original derivation of the differential equations for this third order loop came from Dr. John Y. Huang, of Ford Aerospace.

$$
\begin{aligned}
q(0^+) &= (b_1 - a_1)KK_p \sin\theta_0 + \left.\frac{d\gamma}{dt}\right|_{0^+} \\
&\quad + KK_p \cos\theta_0 \left.\frac{d\theta}{dt}\right|_{0^+} \\
&\quad - KK_p \cos\theta_0 K_v(\gamma(0^+) + KK_p \sin\theta_0) \quad (11)
\end{aligned}
$$

II. Linear Analysis

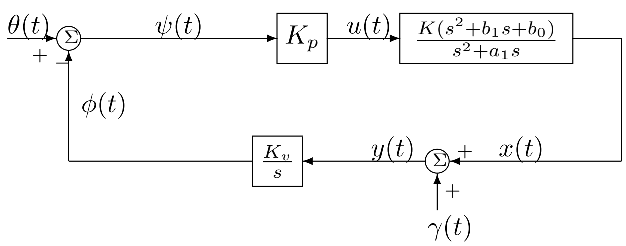

Fig. 2. Linearized Third Order Phase-Lock Loop Block Diagram

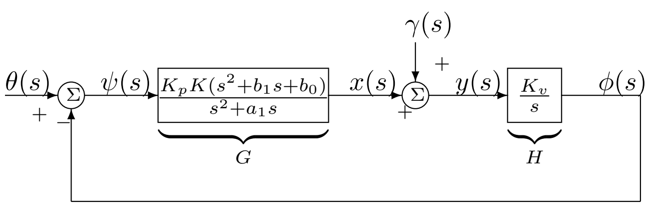

Fig. 3. PLL as a Linear Control System

The only nonlinear equation above is Equation (7). In order to linearize this a small angle assumption is made , *i.e.,*

$$
\begin{aligned}
\psi(t) \text{ small} &\implies \sin\psi(t) \approx \psi(t), && (12)\\
&\implies \cos\psi(t) \approx 1, \text{ and} && (13)\\
\psi(t) \text{ slowly varying} &\implies \left(\frac{d\psi}{dt}\right)^2 \approx 0; \text{ where} && (14)
\end{aligned}
$$

$$
\begin{aligned}
\left(\frac{d\psi}{dt}\right)^2 &= \left(-K_v y + \frac{d\theta}{dt}\right)^2 \\
&= K_v^2 y^2 - 2K_v \frac{d\theta}{dt} y + \left(\frac{d\theta}{dt}\right)^2. \quad (15)
\end{aligned}
$$

Substituting these approximations in (7) yields:

$$
\begin{aligned}
\frac{dq}{dt} &= KK_p b_0 \psi(t) - KK_p K_v b_1 y(t) \\
&\quad -(a_1 + KK_p K_v) q(t) \\
&\quad + KK_p \left\{ \frac{d^2\theta}{dt^2} + b_1 \frac{d\theta}{dt} \right\} + \frac{d^2\gamma}{dt^2} + a_1 \frac{d\gamma}{dt}. \quad (16)
\end{aligned}
$$

The block diagram for this is shown in Figure 2. The initial conditions are computed using Equations (12)–(14), evaluated at $t = 0^+$, in Equations (8)–(11) to yield:

$$
\begin{aligned}
\psi(0^+) &= \theta_0, && (17)\\
x(0^+) &= KK_p\theta_0, && (18)\\
y(0^+) &= \gamma(0^+) + KK_p\theta_0, \text{ and} && (19)\\
q(0^+) &= (b_1 - a_1)KK_p\theta_0 + \left.\frac{d\gamma}{dt}\right|_{0^+} + \\
&\quad KK_p \left\{ \left.\frac{d\theta}{dt}\right|_{0^+} - K_v(\gamma(0^+) + KK_p\theta_0) \right\} && (20)
\end{aligned}
$$

A. Stability of Linear System

With very little effort, Figure 2 can be redrawn in the form of a feedback control system as shown in Figure 3. A result from linear system theory is that the system shown in Figure 3 will be both internally and externally stable if the transfer functions $H_{\psi\theta}$, $H_{\psi\gamma}$, $H_{y\theta}$, and $H_{y\gamma}$ are all stable[5], where

$$
\begin{aligned}
H_{\psi\theta} &= \frac{1}{1+GH} && (21)\\
&= \frac{s^3 + a_1 s}{s^3 + a_1 s^2 + K_p K K_v (s^2 + b_1 s + b_0)}, && (22)\\
H_{\psi\gamma} &= \frac{-H}{1+GH} && (23)\\
&= \frac{-K_v(s^2 + a_1 s)}{s^3 + a_1 s^2 + K_p K K_v (s^2 + b_1 s + b_0)}, && (24)\\
H_{y\theta} &= \frac{G}{1+GH} && (25)\\
&= \frac{K_p K (s^2 + b_1 s + b_0) s}{s^3 + a_1 s^2 + K_p K K_v (s^2 + b_1 s + b_0)}, && (26)
\end{aligned}
$$

and

$$
H_{y\gamma} = \frac{1}{1+GH} = H_{\psi\theta}. \quad (27)
$$

Thus, providing there are no unstable pole-zero cancellations[6], internal and external stability of the linearized PLL depends upon the roots of

$$
\alpha(s) = s^3 + a_1 s^2 + K_p K K_v (s^2 + b_1 s + b_0), \quad (28)
$$

having negative real parts. Routh's stability criterion[7] leads to the following conditions on (28):

$$
\begin{aligned}
a_1 + K_p K K_v &> 0, && (29)\\
K_p K K_v b_1 &> 0, && (30)\\
K_p K K_v b_0 &> 0, \text{ and} && (31)\\
\left[(a_1 + K_p K K_v) b_1 - b_0\right] K_p K K_v &> 0. && (32)
\end{aligned}
$$

Typically, the gains are chosen so that $K_p K K_v > 0$, thus reducing the stability conditions to:

$$
\begin{aligned}
a_1 + K_p K K_v &> 0, && (33)\\
b_1 &> 0, && (34)\\
b_0 &> 0, \text{ and} && (35)\\
(a_1 + K_p K K_v) b_1 &> b_0. && (36)
\end{aligned}
$$

Conditions (34) and (35) require that the open loop filter be minimum phase. Conditions (33) and (36) are essentially conditions on the gain parameters, K_p, K, and K_v.

III. Nonlinear Analysis

The second method of Lyapunov[8] is commonly used in stability analysis of nonlinear differential equations because it does not require the solution to the differential equation. A very intuitive discussion of this can be found in Ogata, [9]. The second method of Lyapunov is based on the generalized energy in the system. If an energy function of the system state is found which is constantly decreasing, then the system is asymptotically stable.[2] A general form of a vector differential equation is:

$$\dot{x} = f(x,t) \quad \text{where} \quad x, \dot{x} \in R^n. \tag{37}$$

An equilibrium state is any state such that

$$f(x_e, t) = 0. \tag{38}$$

Usually, a transformation is made so that the origin of state space is an equilibrium state , *i.e.,*

$$f(0,t) = 0. \tag{39}$$

Theorem 1 (Lyapunov's Main Stability Theorem): For the system defined by Equation 37, suppose there exists a positive definite scalar function of x, $V(x)$, *i.e.,*

$$\begin{array}{rcll} V(x) & > & 0 & \forall x \neq 0 \\ V(x) & = & 0 & x = 0, \end{array} \tag{40}$$

such that $\dot{V}(x)$ is negative definite , *i.e.,*

$$\begin{array}{rcll} \dot{V}(x) & < & 0 & \forall x \neq 0 \\ \dot{V}(x) & = & 0 & x = 0. \end{array} \tag{41}$$

Then (37) is globally asymptotically stable.[8]

Theorem 2 (LaSalle's Theorem): For the system defined by Equation 37, suppose there exists a positive definite scalar function of x, $V(x)$, such that $\dot{V}(x)$ is negative semi-definite , *i.e.,*

$$\begin{array}{rcll} \dot{V}(x) & \leq & 0 & \forall x \neq 0 \\ \dot{V}(x) & = & 0 & x = 0. \end{array} \tag{42}$$

Suppose also that the only solution of $\dot{x} = f(x,t)$, $\dot{V}(x) = 0$ is $x(t) = 0$ for all $t \geq 0$. Then $\dot{x} = f(x,t)$ is globally asymptotically stable.

A. Stability of Nonlinear System

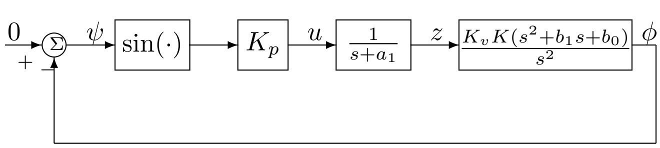

Fig. 4. Closed-Loop Nonlinear System

[2]A generalized energy function is any positive definite function of the system states which is nonvanishing for any state $\neq 0$.

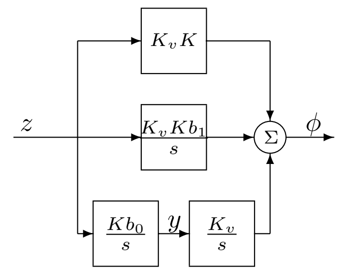

Fig. 5. Drawing out the state variables

Stability analysis is done for the homogeneous (no input) equations, therefore it is convenient to redraw Figure 1 as shown in Figure 4. Furthermore, drawing the map from z to ϕ as shown in Figure 5 will be useful. Note that the internal state variable, y, is defined differently from above because of the regrouping of the integrators. In order to prove the stability of the nonlinear model, the second method of Lyapunov [8] will be used. A Lyapunov function of the kind described by LaSalle and Lefschetz [10] will be used. Finally, LaSalle's Theorem will be invoked proving stability for $\psi \in (-\pi, \pi)$.

The state equations corresponding to Figures 4 and 5 are:

$$\begin{aligned} \dot{z} &= K_p \sin\psi - a_1 z && (43) \\ \dot{y} &= K b_0 z && (44) \\ \dot{\phi} &= K_v K \dot{z} + K_v K b_1 z + K_v y && (45) \\ &= K_v K K_p \sin\psi + K_v K (b_1 - a_1) z + K_v y, && (46) \end{aligned}$$

and

$$\begin{aligned} \dot{\psi} &= -\dot{\phi} && (47) \\ &= -K_v K K_p \sin\psi - K_v K (b_1 - a_1) z \\ &\quad - K_v y. && (48) \end{aligned}$$

Choose

$$V = \int_0^{\psi} \sin(\sigma) d\sigma + \frac{1}{2} \left[\begin{array}{cc} z & y \end{array} \right] P \left[\begin{array}{c} z \\ y \end{array} \right], \tag{49}$$

where

$$P = \left[\begin{array}{cc} p_{11} & p_{12} \\ p_{12} & p_{22} \end{array} \right] \tag{50}$$

is a symmetric, positive definite matrix. Then

$$\begin{aligned} \dot{V} &= \sin(\psi)\dot{\psi} + \left[\begin{array}{cc} z & y \end{array} \right] \left[\begin{array}{cc} p_{11} & p_{12} \\ p_{12} & p_{22} \end{array} \right] \left[\begin{array}{c} \dot{z} \\ \dot{y} \end{array} \right] && (51) \\ &= \sin(\psi)\dot{\psi} + p_{11} z \dot{z} + p_{12} y \dot{z} + p_{12} z \dot{y} + p_{22} y \dot{y} && (52) \\ &= \sin\psi \left[-K_v K K_p \sin\psi - K_v K (b_1 - a_1) z + K_v y \right] \\ &\quad + p_{11} z \left[K_p \sin\psi - a_1 z \right] + p_{12} y \left[K_p \sin\psi - a_1 z \right] \\ &\quad + p_{12} z K b_0 z + p_{22} y K b_0 z && (53) \\ \dot{V} &= -\sin^2\psi \left[K_v K K_p \right] + z^2 \left[p_{12} K b_0 - p_{11} a_1 \right] \\ &\quad + \sin\psi \left\{ z \left(p_{11} K_p - K_v K (b_1 - a_1) \right) \right. \\ &\quad \left. + y \left(p_{12} K_p - K_v \right) \right\} + yz \left[p_{22} K b_0 - p_{12} a_1 \right]. && (54) \end{aligned}$$

In order to invoke LaSalle's Theorem, we must have $V(\psi, y, z) \geq 0$ with $V = 0 \iff \psi = y = z = 0$ and $\dot{V} \leq 0$. Assuming $\psi \in (-\pi, \pi)$ the conditions for $V \geq 0$ are those that guarantee that P is positive definite:

$$\begin{aligned} p_{11} &> 0 \text{ and} && (55)\\ p_{11}p_{22} &> (p_{12})^2 \text{ , which leads to} && (56)\\ p_{22} &> 0. && (57)\\ & && (58)\end{aligned}$$

The conditions that guarantee $\dot{V} \leq 0$ are:

$$\begin{aligned} K_v K K_p &> 0, && (59)\\ p_{12}Kb_0 - p_{11}a_1 &< 0, && (60)\\ p_{11}K_p - K_v K(b_1 - a_1) &= 0, && (61)\\ p_{12}K_p - K_v &= 0, \quad \text{and} && (62)\\ p_{22}Kb_0 - p_{12}a_1 &= 0. && (63)\end{aligned}$$

Conditions 62 and 61 imply

$$p_{12} = \frac{K_v}{K_p} \qquad (64)$$

and

$$p_{11} = \frac{K_v K}{K_p}(b_1 - a_1), \qquad (65)$$

respectively. From Condition 59 we get

$$K_v K K_p > 0 \Leftrightarrow \frac{K_v K K_p}{K_p^2} > 0 \Leftrightarrow \frac{K_v K}{K_p} > 0, \qquad (66)$$

and using Condition 55 yields

$$b_1 > a_1. \qquad (67)$$

From Condition 63 and Equation 64 we get

$$p_{22} = \frac{K_v a_1}{K_p K b_0}. \qquad (68)$$

Condition 60 implies

$$\begin{aligned} \frac{K_v K}{K_p} b_0 - \frac{K_v K}{K_p}(b_1 - a_1)a_1 &< 0 \text{ or} && (69)\\ \frac{K_v K}{K_p}\left[b_0 - (b_1 - a_1)a_1\right] &< 0. && (70)\end{aligned}$$

Equation 66,

$$\frac{K_v K}{K_p} > 0, \qquad (71)$$

implies

$$b_0 - \underbrace{(b_1 - a_1)}_{>0} a_1 < 0. \qquad (72)$$

Condition 57 requires that

$$p_{22} = \frac{K_v a_1}{K_p K b_0} > 0, \qquad (73)$$

which can be assured using (66) and by assuming that b_0 and a_1 have the same sign. It is convenient to choose both b_0 and $a_1 > 0$. Under the above assumptions, Condition 56 is completely equivalent to Condition 60.

In summary, choose K_p, K, K_v, a_1, b_0, and $b_1 > 0$. Then choose

$$\begin{aligned} b_1 &> a_1 \text{ and} && (74)\\ b_0 - (b_1 - a_1)a_1 &< 0. && (75)\end{aligned}$$

Then the above choice of

$$P = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \quad \text{implies} \qquad (76)$$

$$\begin{aligned} \dot{V} &= -\sin^2\psi\left[K_v K K_p\right] \\ &\quad + z^2\left[\frac{K_v K}{K_p}(b_0 - (b_1 - a_1)a_1)\right] && (77)\\ &\leq 0. && (78)\end{aligned}$$

Finally, the only place that $\dot{V}$ and (43) – (48) can vanish is for $z = y = \phi = \psi = 0$, so using LaSalle's Theorem proves stability.

IV. Conclusions

Lyapunov stability techniques are adequate for analyzing the stability of a third order phase-lock loop is not substantially more difficult than that of a second order loop. We have developed both a linear and a nonlinear model of a third order phase-lock loop and have developed stability conditions for both. It goes without saying that it should be possible to apply these same techniques to all orders of phase-lock loops, as will be shown in [11].

V. Acknowledgements

The author would like to thank Daniel Witmer of Ford Aerospace for introducing the phase-lock loop to him as a nonlinear feedback system. The author would further like to thank Professors Gene Franklin and Stephen Boyd of Stanford University and Dr. Michael Workman of IBM Corporation for introducing him to the techniques that were used to analyze this problem. Also, the author would like to thank Professor Thomas Cover of Stanford University for stressing the importance of looking for unifying principles in all of ones work.

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